The Physical Meaning of Gauge Transformations in Electrodynamics

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Abstract:

The structure of classical electrodynamics based on the variational principle together with causality and space-time homogeneity is analyzed. It is proved that in this case the 4-potentials are defined uniquely. On the other hand, the approach where Maxwell equations and the Lorentz law of force are regarded as cornerstones of the theory allows gauge transformations. For this reason, the two theories are *not equivalent*. A simple example substantiates this conclusion. Quantum physics is linked to the variational principle and it is proved that the same result holds for it. The compatibility of this result with the gauge invariance of the Lagrangian density is explained.

1. Introduction

One may regard the equations of motion of a physical system as the fundamental elements of a theory. Thus, the equations of motion can be used for deriving useful formulas that describe properties of the system. However, it is now recognized that other principles play a more profound role. Using this approach, the variational principle, causality and homogeneity of space-time are regarded here as the basis for the discussion. The present work examines these approaches within the validity domains of classical electrodynamics and of quantum physics. Thus, the electrodynamic theory that regards Maxwell equations and the Lorentz law of force as cornerstones of the theory is called here Maxwell-Lorentz electrodynamics (MLE). The theory that relies on the variational principle is called here canonical electrodynamics (CE). MLE and CE are very closely related theories. Thus, Maxwell equations and the Lorentz law of force can be derived from the variational principle (see [1], pp. 49-51,70,71,78-80; [2], 572-578,595-597). The first part of the discussion carried out here analyzes the two approaches within the realm of classical electrodynamics and proves that MLE is *not equivalent* to CE and that CE imposes further restrictions on the theory's structure. Quantum mechanics is strongly linked to the variational approach (see [3], pp. 2-23). Thus, it is proved that the same results are valid for quantum mechanics.

It is proved in this work that if one adheres to CE together with causality and space-time homogeneity then the 4-potentials of electrodynamics are defined uniquely. On the other hand, the 4-potentials play no explicit role in Maxwell equations and in the Lorentz law of force. Hence, one may apply any gauge transformation without affecting MLE. This is the underlying reason

for the claim that MLE is *not equivalent* to CE.

In the present work, units where the speed of light $c = 1$ and $\hbar = 1$ are used. Thus, one kind of dimension exists and the length $[L]$ is used for this purpose. Greek indices run from 0 to 3. The metric is diagonal and its entries are (1,-1,-1,-1). The symbol $_{,\mu}$ denotes the partial differentiation with respect to x^μ. A_μ denotes the 4-potentials and $F^{\mu\nu}$ denotes the antisymmetric tensor of the electromagnetic fields

$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}(A_{\beta,\alpha} - A_{\alpha,\beta}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{1}$$

In the second Section, the main point of this work is proved for classical physics. The third Section describes a specific example that substantiates the proof included in Section 2. The fourth Section proves that the same results are obtain for quantum physics. The last Section contains concluding remarks.

2. Gauge Transformations and Canonical Electrodynamics

The Lagrangian density used for a derivation of Maxwell equations is (see [1], pp. 78-80; [2], pp. 596-597)

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu. \tag{2}$$

The following analysis examines a closed system of charges and fields. For the simplicity of the discussion, let us examine the fields associated with one charged particle e whose motion is given. This approach can be justified because, due to the linearity of Maxwell equations, one finds that the fields of a closed system of charges is a superposition of the fields of each individual

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charge belonging to the system. Let us examine the electromagnetic fields at a given space-time point x^μ. Using Maxwell equation and the principle of causality, one can derive the retarded Lienard-Wiechert 4-potentials (see [1], pp. 173-174; [2], pp. 654-656)

$$A_\mu = e\frac{v_\mu}{R^\alpha v_\alpha}. \tag{3}$$

Here v_μ is the charge's 4-velocity at the retarded time and R^μ is the 4-vector from the retarded space-time point to the field point x^μ. These 4-potentials defines the fields uniquely.

A gauge transformation of (3) is (see [1], pp. 52-53; [2], pp. 220-223)

$$A'_\mu = A_\mu - \Phi_{,\mu}. \tag{4}$$

In the following lines, the laws of CE are used in an investigation of the form of the gauge function $\Phi(x^\mu)$.

Relying on the variational principle, one finds constraints on terms of the Lagrangian density. Thus, the action is a Lorentz scalar and in the unit system used here where $\hbar = 1$, it is dimensionless. In particular, the 4-potentials A_μ must be entries of a 4-vector whose dimension is $[L^{-1}]$. This requirement is satisfied by the Lienard-Wiechert 4-potentials (3). Thus, also $\Phi_{,\mu}$ of (4) is a 4-vector and Φ must be a dimensionless Lorentz scalar function of the space-time coordinates.

Now, the coordinates are entries of a 4-vector. Therefore, a Lorentz scalar depending on the space-time coordinates must be a function of scalar variables of the following form

$$f_{a,b}(x^\mu) = (x^\mu - x_a^\mu)(x_\mu - x_{b\mu}), \tag{5}$$

where x_a^μ and x_b^μ denote specific space-time points. Relying on the homogeneity of space-time, one finds that in the case discussed here there is just

one specific point x_a^μ, which is the retarded position of the charge. Thus, (5) is cast into the following form

$$f_{a,b}(x^\mu) \rightarrow R^\mu R_\mu. \tag{6}$$

This outcome proves that the gauge function, which is a dimensionless quantity, must be a constant.

These arguments complete the proof showing that if one adheres to CE then the gauge function Φ is a constant and the gauge 4-vector $\Phi_{,\mu}$ vanishes identically. Hence, the Lienard-Wiechert 4-vector (3) is unique.

3. An Example

Let us examine a simple system which consists of one motionless particle whose mass and charge are m, e, respectively. The particle is located in a spatial region where the external fields vanish. Therefore, the Lorentz force exerted on the particle vanishes too and it remains motionless as long as these conditions do not change. Hence, from the point of view of MLE, the particle's energy is a constant

$$E = m. \tag{7}$$

Now, let us examine this system from the point of view of CE. For this purpose, the external 4-potentials should be defined. Thus, the null external fields are derived from null 4-potentials

$$A_{(ext)\mu} = 0 \quad \rightarrow \quad F^{\mu\nu}_{(ext)} = 0. \tag{8}$$

In order to define the particle's energy one must construct the Hamiltonian. Here the general expression is (see [1], pp. 47-49; [2], pp. 575)

$$H = [m^2 + (\mathbf{P} - e\mathbf{A})^2]^{1/2} + e\phi, \tag{9}$$

where $\mathbf{P}$ denotes the canonical momentum and the components of the 4-potentials are $(\phi, \mathbf{A})$. Substituting the null values of (8) into (9) and putting there $\mathbf{P} = 0$ for the motionless particle, one equates the energy to the Hamiltonian's value and obtains

$$E = m. \tag{10}$$

At this point, one finds that result (7) of MLE is identical to (10) of CE.

Now, let us apply a gauge transformation to the null external 4-potentials (8). The gauge function and its 4-potentials are

$$\Phi = t^2 \quad \rightarrow \quad A'_{(ext)\mu} = -\Phi_{,\mu} = (-2t, 0, 0, 0). \tag{11}$$

In MLE nothing changes, because the equations of motion depend on electromagnetic fields and their null value does not change

$$F'^{\mu\nu} = F^{\mu\nu} = 0. \tag{12}$$

Hence, the energy value (7) continues to hold and the gauge transformation (11) is acceptable in MLE.

The following points show several arguments proving that this conclusion does not hold for the CE theory.

1. The gauge function of (11) has the dimensions $[L^2]$, whereas in CE it must be dimensionless.

2. The gauge function of (11) is the entry U^{00} of the second rank tensor $U^{\mu\nu} = x^\mu x^\nu$. On the other hand, in CE the gauge function must be a Lorentz scalar.

3. Substituting the gauge 4-vector $A'_{(ext)\mu}$ of (11) into the Hamiltonian (9), one finds the following value for the energy

$$E' = H' = m - 2et. \tag{13}$$

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Hence, if gauge transformations are allowed in CE then the energy of a closed system is not a constant of the motion.

These three conclusions prove that a gauge transformation destroys CE.

4. Gauge Transformations and Quantum Physics

As stated in the Introduction, quantum physics is very closely related to CE. Moreover, the Ehrenfest theorem (see [5], pp. 25-27, 138) shows that the classical limit of quantum mechanics agrees with the laws of classical physics. For these reasons, one expects that the laws of CE are relevant to quantum physics. A direct examination of gauge transformations proves this matter.

The Lagrangian density of the Dirac field is (see [3], p. 84; [4], p. 78)

$$\mathcal{L} = \bar{\psi}[\gamma^{\mu}(i\partial_{\mu} - eA_{\mu}) - m]\psi, \tag{14}$$

Now, in quantum mechanics, the gauge transformation (4) is accompanied by an appropriate transformation of the particle's wave function. Thus, the quantum mechanical form of gauge transformation is (see [4], p. 78)

$$A'_{\mu} = A_{\mu} - \Phi_{,\mu}; \quad \psi'(x^{\mu}) = e^{ie\Phi(x^{\mu})}\psi(x^{\mu}) \tag{15}$$

(Note that the symbol e in the exponent denotes the particle's electric charge.) Substituting the gauge transformation (15) into the Lagrangian density (14), one realized the it is gauge invariant indeed (see e.g. [4], p. 78).

Now let us examine the quantum mechanical version of the example discussed in Section 3. The Dirac wave function of the spin-up state of a motionless particle is (see [6], p. 10)

$$\psi(x^{\mu}) = e^{-imt}(1, 0, 0, 0). \tag{16}$$

Thus, one uses the fundamental quantum mechanical equation and obtains the particle's energy from an application of the Dirac Hamiltonian to the wave function (16)

$$E\psi = H\psi = i\frac{\partial\psi}{\partial t} = m\psi \rightarrow E = m. \tag{17}$$

Now, let us examine the gauge transformation (15) for the specific case (11). The wave function (16) transforms as follows

$$\psi'(x^\mu) = e^{iet^2}e^{-imt}(1,0,0,0). \tag{18}$$

A straightforward calculation of the energy for the gauge transformed wave function (18) proves that the result differs from the original value

$$E'\psi' = i\frac{\partial\psi'}{\partial t} = (m - 2et)\psi' \rightarrow E' = m - 2et. \tag{19}$$

This is precisely the same discrepancy which was found above for the gauge transformation of CE of classical physics (13). Thus, one concludes that gauge transformations are inconsistent with quantum physics.

5. Conclusions

The foregoing results indicate the difference between an electrodynamic theory where Maxwell equations and the Lorentz law of force are regarded as the theory's cornerstones and a theory based on the variational principle together with causality and space-time homogeneity. Indeed, if Maxwell equations and the Lorentz law of force are the theory's cornerstone then it is very well known that one is free to define the gauge function $\Phi(x^\mu)$ of (4) (see [1], pp. 52-53; [2], pp. 220-223). On the other hand, this work proves that gauge transformations are inconsistent with electrodynamics based on the

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variational principle. For this reason, one concludes that the two approaches are *not equivalent.* It is also proved that gauge transformations are forbidden in quantum physics.

The outcome of this work does not negate the well known gauge invariance of the Lagrangian density. Indeed, in the Dirac Lagrangian density (14), the two parts of the gauge transformation (15) cancel each other. On the other hand, the Dirac Hamiltonian contains only one term of (15).

References:

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